NEW UPPER AND LOWER BOUND SIFTING ITERATIONS

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1. Introduction

Let A be a (possibly weighted) set of whole numbers, and for each positive integer d set $A_d = \{a \in A, d \mid a\}$. Suppose that κ, z, y are such that for every squarefree integer d, all of whose prime factors are less than z, we have

$$\left| |A_d| - \kappa^{\omega(d)} \frac{y}{d} \right| \le 1. \tag{1}$$

In particular, we have $y - 1 \le |A| \le y + 1$. We want to estimate the quantity

$$\mathcal{S}(A, z) = |\{a \in A, \forall p < z \ (a, p) = 1\}| \, .$$

Suppose now that $y = z^s$, s a constant, y, z going to infinity. Define sifting functions $f_\kappa(s), F_\kappa(s)$ by

$$(1 + o(1)) f_\kappa(s) y \prod_{p<z} \left(1 - \frac{\kappa}{p}\right) \le \mathcal{S}(A, z) \le (1 + o(1)) F_\kappa(s) y \prod_{p<z} \left(1 - \frac{\kappa}{p}\right),$$

with $f_\kappa(s)$ as large as possible (resp. $F_\kappa(s)$ as small as possible) given that the above inequality holds for all choices of A satisfying (1). Selberg [3] has shown (in a much more general context) that the functions $f_\kappa(s), F_\kappa(s)$ are continuous, monotone, and computable for $s > 1$, and that they tend to 1 exponentially as s goes to infinity.

Let $\beta = \beta(\kappa)$ be the infimum of s such that $f_\kappa(s) > 0$. Selberg [3] has shown that we have

$$\frac{\kappa}{e^{(1-\frac{1}{\kappa})^2}} < \beta < 2\kappa + 0.4454,$$

where the first inequality applies for all $\kappa \ge 1$ and the second applies for κ sufficiently large. Further, when $\kappa = 1$ we have $\beta(1) = 2$. Our main goal is to get good estimates for $\beta(\kappa)$ when κ is slightly greater than 1.

When $\kappa \le 1$, the best known sieves arise from the identity

$$\mathcal{S}(A, z) = |A| - \sum_{p<z} \mathcal{S}(A_p, p),$$

p running over primes. This identity leads to the inequalities

$$s^\kappa f_\kappa(s) \ge s^\kappa - \kappa \int_{t>s} t^{\kappa-1}(F_\kappa(t-1) - 1)dt,$$
$$s^\kappa F_\kappa(s) \le s^\kappa + \kappa \int_{t>s} t^{\kappa-1}(1 - f_\kappa(t-1))dt.$$

The use of these inequalities to produce new bounds on f_κ, F_κ from known bounds is known as Buchstab iteration (reference?). Infinite iteration of Buchstab's inequalities leads to what is known as the β-sieve.

The current state of the art for κ slightly greater than 1 is due to Diamond, Halberstam, and Richert [1]. Their method runs as follows. From Selberg's upper bound sieve, we have

$$F_\kappa(s) \le \frac{1}{\sigma_\kappa(s)},$$

where $\sigma_\kappa(s)$ solves the differential-difference equation

$$\begin{cases} s^{-\kappa}\sigma_\kappa(s) = \frac{1}{(2e^\gamma)^\kappa\Gamma(\kappa+1)} & 0 < s \le 2, \\ \frac{d}{ds}(s^{-\kappa}\sigma_\kappa(s)) = -\kappa s^{-\kappa-1}\sigma(s-2) & 2 \le s. \end{cases}$$

Using this as our starting point, we iterate Buchstab's inequalities infinitely to obtain upper and lower bounds f_κ^D, F_κ^D satisfying

$$\begin{aligned} F_\kappa^D(s) &= \frac{1}{\sigma_\kappa(s)} & 0 < s \le \alpha_\kappa^D, \\ \frac{d}{ds}\left(s^\kappa F_\kappa^D(s)\right) &= \kappa s^{\kappa-1} f_\kappa^D(s-1) & s \ge \alpha_\kappa^D, \\ f_\kappa^D(s) &= 0 & 0 < s \le \beta_\kappa^D, \\ \frac{d}{ds}\left(s^\kappa f_\kappa^D(s)\right) &= \kappa s^{\kappa-1} F_\kappa^D(s-1) & s \ge \beta_\kappa^D, \end{aligned}$$

for some $\alpha_\kappa^D \ge \beta_\kappa^D$.

In this note we'll describe variations on Buchstab's inequalities which allow us to slightly improve upon the functions f_κ^D, F_κ^D above.

2. The new iterations

Theorem 1. *For any $w \le z$, we have*

$$\mathcal{S}(A,z) \le \mathcal{S}(A,w) - \frac{2}{3}\sum_{w \le p < z} \mathcal{S}(A_p, w) + \frac{1}{3}\sum_{w \le q < p < z} \mathcal{S}(A_{pq}, w),$$

where p, q run over primes.

Proof. Let $a \in A$. We need to show that the number of times a is counted on the left hand side of the above is at least the number of times a is counted on the right. If a has any prime factors below w, then both quantities are clearly zero, so assume that a is has no prime factors below w. Suppose a has exactly k prime factors between w and z. If $k = 0$ then both sides count a once. Thus we just need to check that for any integer $k \ge 1$ we have

$$0 \le 1 - \frac{2}{3}k + \frac{1}{3}\binom{k}{2},$$

which follows from the identity

$$1 - \frac{2}{3}k + \frac{1}{3}\binom{k}{2} = \left(1 - \frac{k}{2}\right)\left(1 - \frac{k}{3}\right). \qquad \square$$

Corollary 1. *For any real $t \ge s \ge 2$, we have*

$$s^\kappa F_\kappa(s) \le t^\kappa F_\kappa(t) - \frac{2}{3}\kappa \int\limits_{\frac{1}{t} < x < \frac{1}{s}} t^\kappa f_\kappa(t(1-x))\frac{dx}{x} + \frac{1}{3}\kappa^2 \iint\limits_{\frac{1}{t} < y < x < \frac{1}{s}} t^\kappa F_\kappa(t(1-x-y))\frac{dx}{x}\frac{dy}{y}.$$

Remark 1. The optimal w in Theorem 1 above appears to be $w = \frac{y}{z^\beta}$, which corresponds to taking $t = \frac{s}{s-\beta}$. Thus this upper bound iteration tends to be useful only for $2 \le s \le \beta + 1$.

Theorem 2. *For any $w \le z^2$, we have*

$$\mathcal{S}(A,z) \ge \mathcal{S}\Big(A,\sqrt{w}\Big) - \sum_{\sqrt{w}\le p<z} \mathcal{S}\Big(A_p,\frac{w}{p}\Big) + \frac{5}{6}\sum_{\frac{w}{p}\le q<p<z} \mathcal{S}\Big(A_{pq},\frac{w}{p}\Big)$$
$$-\frac{2}{3}\sum_{\substack{\frac{w}{p}\le r<q<p<z\\ qr<w}} \mathcal{S}\Big(A_{pqr},\frac{w}{p}\Big) - \frac{1}{2}\sum_{\frac{w}{q}\le r<q<p<z} \mathcal{S}\Big(A_{pqr},\frac{w}{p}\Big),$$

where p,q,r run over primes.

Proof. Let $a \in A$. First suppose that a has no prime factors below $\sqrt{w}$, and has exactly k prime factors between $\sqrt{w}$ and z. If k is 0, then both sides count a once. Otherwise, we need to check that for an integer $k \ge 1$ we have

$$0 \ge 1 - k + \frac{5}{6}\binom{k}{2} - \frac{1}{2}\binom{k}{3},$$

and this follows from the identity

$$1 - k + \frac{5}{6}\binom{k}{2} - \frac{1}{2}\binom{k}{3} = (1-k)\left(1-\frac{k}{3}\right)\left(1-\frac{k}{4}\right).$$

Now suppose that a has smallest prime factor $s < \sqrt{w}$. We group together all of the summands on the right hand side with a common p, $p \mid a$. In order for any such summand to be nonzero, we must have $s \ge \frac{w}{p}$, or equivalently $p \ge \frac{w}{s}$. Suppose that a has exactly k prime factors strictly below p. Then the number of times a is counted in such summands is at most

$$-1 + \frac{5}{6}k - \frac{1}{2}\binom{k}{2} = -\left(1-\frac{3k}{4}\right)\left(1-\frac{k}{3}\right),$$

and this is at most 0 unless $k = 2$. Thus the only bad case occurs when p is the third smallest prime factor of a, q is the second smallest prime factor of a, and $r = s$ is the smallest prime factor of a. If $qr < w$, then the contribution from these summands is just

$$-1 + \frac{5}{6}\cdot 2 - \frac{2}{3} = 0,$$

so the bad case only occurs when $qr \ge w$. But then since $q \ge \frac{w}{r} = \frac{w}{s}$, we can combine this bad group of summands with the group of summands where p is replaced by q, and the total number of times a is counted in the two groups becomes

$$\left(-1 + \frac{5}{6}\cdot 2 - \frac{1}{2}\right) + \left(-1 + \frac{5}{6}\right) = \frac{1}{6} - \frac{1}{6} = 0. \qquad \square$$

Corollary 2. *For any real $s \geq t$ with $2t \geq s \geq 3$, we have*

$$
\begin{aligned}
s^\kappa f_\kappa(s) \geq (2t)^\kappa f_\kappa(2t) &- \kappa \int\limits_{\frac{1}{2t}<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} F_\kappa\Big(\frac{1-x}{\frac{1}{t}-x}\Big)\frac{dx}{x} \\
&+ \frac{5}{6}\kappa^2 \iint\limits_{\frac{1}{t}-x<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} f_\kappa\Big(\frac{1-x-y}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y} \\
&- \frac{2}{3}\kappa^3 \iiint\limits_{\frac{1}{t}-x<z<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} F_\kappa\Big(\frac{1-x-y-z}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z} \\
&+ \frac{1}{6}\kappa^3 \iiint\limits_{\frac{1}{t}-y<z<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^\kappa} F_\kappa\Big(\frac{1-x-y-z}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}.
\end{aligned}
$$

Remark 2. As with Theorem 1, it seems that the optimal w in Theorem 2 is $w = \frac{y}{z^\beta}$, corresponding to $t = \frac{s}{s-\beta}$. Thus this lower bound iteration tends to be useful only when $\beta + 1 \leq s \leq \beta + 2$.

3. Two miracles at $\kappa = 1$

When $\kappa = 1$, the β-sieve produces the optimal functions $f(s) = f_1(s), F(s) = F_1(s)$ (see Selberg [3]). Furthermore, we have the more precise error terms

$$f(s)\frac{y}{e^\gamma \log(z)} - (c+o(1))h(s)\frac{y}{\log(z)^2} \leq \mathcal{S}(A,z) \leq F(s)\frac{y}{e^\gamma \log(z)} + (c+o(1))H(s)\frac{y}{\log(z)^2},$$

where c is a computable constant (in fact a more precise result can be found in Iwaniec [2]). The functions f, F, h, H are given by

$$
\begin{aligned}
F(s) &= \frac{2e^\gamma}{s} && 1 \leq s \leq 3 \\
\frac{d}{ds}\left(sF(s)\right) &= f(s-1) && s \geq 3 \\
f(s) &= \frac{2e^\gamma \log(s-1)}{s} && 2 \leq s \leq 4 \\
\frac{d}{ds}\left(sf(s)\right) &= F(s-1) && s \geq 2 \\
H(s) &= \frac{1}{s^2} && 1 \leq s \leq 3 \\
\frac{d}{ds}\left(s^2H(s)\right) &= -sh(s-1) && s \geq 3 \\
h(s) &= \frac{1}{s^2}\left(1+\frac{1}{s-1}-\log(s-1)\right) && 2 \leq s \leq 4 \\
\frac{d}{ds}\left(s^2h(s)\right) &= -sH(s-1) && s \geq 2
\end{aligned}
$$

It's natural to ask what happens to these functions when we apply the new upper and lower bound iterations to them.

Theorem 3. *If $\kappa = 1$, $\frac{5}{2} \le s \le 3$, and $t = \frac{s}{s-2}$, then the two sides of the inequality in Corollary 1 are precisely equal, that is*

$$sF(s) = tF(t) - \frac{2}{3} \int\limits_{\frac{1}{t}<x<\frac{1}{s}} tf(t(1-x))\frac{dx}{x} + \frac{1}{3} \iint\limits_{\frac{1}{t}<y<x<\frac{1}{s}} tF(t(1-x-y))\frac{dx}{x}\frac{dy}{y}.$$

Furthermore, in this case even the error terms match up:

$$s^2H(s) = t^2H(t) + \frac{2}{3} \int\limits_{\frac{1}{t}<x<\frac{1}{s}} t^2h(t(1-x))\frac{dx}{x} + \frac{1}{3} \iint\limits_{\frac{1}{t}<y<x<\frac{1}{s}} t^2H(t(1-x-y))\frac{dx}{x}\frac{dy}{y}.$$

Proof. Consider the right hand side of the first claimed equality as a function $\Phi(s,t)$ of s and t. Since $sF(s) = 2e^\gamma$ is constant for $s \le 3$, it's enough to check that $\frac{\partial\Phi}{\partial s} = \frac{\partial\Phi}{\partial t} = 0$ when $t = \frac{s}{s-2}$. We have

$$\frac{\partial\Phi}{\partial s} = \frac{2}{3}\frac{t}{s}f\left(t\left(1-\frac{1}{s}\right)\right) - \frac{1}{3}\int\limits_{\frac{1}{t}<x<\frac{1}{s}} \frac{t}{s}F\left(t\left(1-\frac{1}{s}-x\right)\right)\frac{dx}{x},$$

and up to a multiple of $\frac{2e^\gamma}{s-1}$ this is equal to

$$\begin{aligned}&\frac{2}{3}\log\left(t\left(1-\frac{1}{s}\right)-1\right) - \frac{1}{3}\left(\log\left(t-\frac{s}{s-1}\right) - \log\left(s-\frac{s}{s-1}\right)\right)\\ &= \frac{1}{3}\log\left(t\frac{s-1}{s}-1\right) + \frac{1}{3}\log(s-2),\end{aligned}$$

which is indeed 0 when $t = \frac{s}{s-2}$. In order to calculate $\frac{\partial\Phi}{\partial t}$, first note that since $\frac{5}{2} \le s$ we have $t = \frac{s}{s-2} \le 5$, so for any $x, y > \frac{1}{t}$ we have $t(1-x-y) \le t-2 \le 3$, so

$$\frac{\partial}{\partial t}\left(tF(t(1-x-y))\right) = 0.$$

Thus we have

$$\frac{\partial\Phi}{\partial t} = f(t-1) - \frac{2}{3}f(t-1) - \frac{2}{3}\int\limits_{\frac{1}{t}<x<\frac{1}{s}} F(t(1-x)-1)\frac{dx}{x} + \frac{1}{3}\int\limits_{\frac{1}{t}<x<\frac{1}{s}} F(t(1-x)-1)\frac{dx}{x} + 0,$$

and up to a multiple of $\frac{2e^\gamma}{t-1}$ this is equal to

$$\begin{aligned}&\frac{1}{3}\log(t-2) - \frac{1}{3}\left(\log\left(t-\frac{t}{t-1}\right) - \log\left(s-\frac{t}{t-1}\right)\right)\\ &= \frac{1}{3}\log\left(s\frac{t-1}{t}-1\right),\end{aligned}$$

which is also equal to 0 when $t = \frac{s}{s-2}$.

The second claim is left as an involved exercise to the reader (alternatively, one can use the method of proof of the next theorem). □

Since the lower bound iteration is much more complicated, we need a better method of checking that it has the linear sieve as a fixed point. For this we use the following weighted sets, introduced by Selberg [3] in order to explain the parity problem: let A^+ be the weighted set of integers between 1 and y with the weight attached to n given by $1 - \lambda(n)$, where $\lambda(n) = (-1)^{\Omega(n)}$, and let A^- be similar with the weight of n given by $1 + \lambda(n)$. Set

$$\pi^\pm(y,z) = \mathcal{S}(A^\pm, z).$$

These functions are invariant under Buchstab iteration:

$$\pi^{\pm}(y,z) = \pi^{\pm}(y,w) - \sum_{w<p<z} \pi^{\mp}(y/p,p),$$

and by the prime number theorem, for $1 < s < 3$ we have

$$\pi^{+}(y,z) = 2(\pi(y)-\pi(z)) = \frac{2e^{\gamma}}{s}\frac{y}{e^{\gamma}\log(z)} + \frac{2}{s^2}\frac{y}{\log(z)^2} + O\left(\frac{y}{\log(z)^3}\right),$$

so for all $s > 1$ we have

$$\pi^{+}(y,z) = F(s)\frac{y}{e^{\gamma}\log(z)} + 2H(s)\frac{y}{\log(z)^2} + O\left(\frac{y}{\log(z)^3}\right), \tag{2}$$

$$\pi^{-}(y,z) = f(s)\frac{y}{e^{\gamma}\log(z)} - 2h(s)\frac{y}{\log(z)^2} + O\left(\frac{y}{\log(z)^3}\right). \tag{3}$$

Theorem 4. *If $\kappa = 1$, $\frac{7}{2} \le s \le 4$, and $t = \frac{s}{s-2}$, then the two sides of the inequality in Corollary 2 are equal, that is*

$$\begin{aligned}
sf(s) = 2tf(2t) &- \int\limits_{\frac{1}{2t}<x<\frac{1}{s}} \frac{1}{\frac{1}{t}-x}F\Big(\frac{1-x}{\frac{1}{t}-x}\Big)\frac{dx}{x}\\
&+ \frac{5}{6}\iint\limits_{\frac{1}{t}-x<y<x<\frac{1}{s}} \frac{1}{\frac{1}{t}-x}f\Big(\frac{1-x-y}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\\
&- \frac{2}{3}\iiint\limits_{\frac{1}{t}-x<z<y<x<\frac{1}{s}} \frac{1}{\frac{1}{t}-x}F\Big(\frac{1-x-y-z}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\\
&+ \frac{1}{6}\iiint\limits_{\frac{1}{t}-y<z<y<x<\frac{1}{s}} \frac{1}{\frac{1}{t}-x}F\Big(\frac{1-x-y-z}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}.
\end{aligned}$$

Furthermore, the error terms are equal as well:

$$\begin{aligned}
s^2h(s) = (2t)^2h(2t) &+ \int\limits_{\frac{1}{2t}<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^2}H\Big(\frac{1-x}{\frac{1}{t}-x}\Big)\frac{dx}{x}\\
&+ \frac{5}{6}\iint\limits_{\frac{1}{t}-x<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^2}h\Big(\frac{1-x-y}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\\
&+ \frac{2}{3}\iiint\limits_{\frac{1}{t}-x<z<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^2}H\Big(\frac{1-x-y-z}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}\\
&- \frac{1}{6}\iiint\limits_{\frac{1}{t}-y<z<y<x<\frac{1}{s}} \frac{1}{(\frac{1}{t}-x)^2}H\Big(\frac{1-x-y-z}{\frac{1}{t}-x}\Big)\frac{dx}{x}\frac{dy}{y}\frac{dz}{z}.
\end{aligned}$$

Proof. By equations (2), (3), it's enough to check that for constant $\frac{7}{2} < s < 4$ and $w = \frac{y}{z^2}$ we have

$$\mathcal{S}(A^-, z) = \mathcal{S}\Big(A^-, \sqrt{w}\Big) - \sum_{\sqrt{w} \le p < z} \mathcal{S}\Big(A_p^+, \frac{w}{p}\Big) + \frac{5}{6} \sum_{\frac{w}{p} \le q < p < z} \mathcal{S}\Big(A_{pq}^-, \frac{w}{p}\Big)$$

$$- \frac{2}{3} \sum_{\substack{\frac{w}{p} \le r < q < p < z \\ qr < w}} \mathcal{S}\Big(A_{pqr}^+, \frac{w}{p}\Big) - \frac{1}{2} \sum_{\frac{w}{q} \le r < q < p < z} \mathcal{S}\Big(A_{pqr}^+, \frac{w}{p}\Big) + O\left(\frac{y}{\log(z)^3}\right).$$

We have the easy inequality $z > \sqrt{w} > y^{3/14}$, and for $\sqrt{w} < p < z$ we have $\frac{w}{p} > \frac{w}{z} > y^{1/7}$ as well as $p\left(\frac{w}{p}\right)^5 > \frac{w^5}{z^4} > y$. Thus if n is a number below y which is counted by either side, then every prime factor of n must be at least $y^{1/7}$, and $\Omega(n)$ must be an even number strictly below $\max(\frac{14}{3}, 1+5) = 6$.

We need to estimate the number of ns below y which contribute more to the left and side than the right hand side. Since the number of nonsquarefree ns which can contribute to either side is at most $3y^{6/7}$, we can assume without loss that n is square free. If $n = pq$ with $p > q$ primes, we must have $z > p$ in order for n to contribute more to the left side than the right side. The number of such n is at most $z^2 < y^{4/7}$, so we may assume without loss that n has four distinct prime factors $p > q > r > s$, at least one of which is below z (so n isn't counted on the left hand side at all).

First consider the case $s \ge \sqrt{w}$. Since $n \le wz^2$, we have $z > q$. Then if n has $3 \le k \le 4$ prime factors below z, n is counted on the right hand side with multiplicity $1 - k + \frac{5}{6}\cdot\binom{k}{2} - \frac{2}{3}\cdot 0 - \frac{1}{2}\cdot\binom{k}{3} = (1-k)\left(1-\frac{k}{3}\right)\left(1-\frac{k}{4}\right) = 0$, so we get the same contribution to both sides.

Now suppose that $s < \sqrt{w}, rs \ge w$. Since $n \le wz^2$, we have $z > q$. Then if n has $3 \le k \le 4$ prime factors below z, n is counted on the right hand side with multiplicity $0 - (k-1) + \frac{5}{6}\cdot\binom{k}{2} - \frac{2}{3}\cdot 0 - \frac{1}{2}\cdot\binom{k}{3} = (1-k)\left(1-\frac{k}{3}\right)\left(1-\frac{k}{4}\right) = 0$, as before.

Next suppose that $w > rs$ and $p > z$. We must have $z > q > \frac{w}{s}$ in order to get any contribution from n. Then n is counted on the right hand side with multiplicity $0 - 1 + \frac{5}{6}\cdot 2 - \frac{2}{3}\cdot 1 - \frac{1}{2}\cdot 0 = 0$, so we get the same contribution from both sides.

Thus any bad n must have $z \ge p > q$ and $w > rs$, $r > s > y^{1/7}$. The number of such n is at most $O\left(\frac{z}{\log(z)}\frac{z}{\log(z)}\frac{w}{\log(w)}\right) = O\left(\frac{y}{\log(z)^3}\right)$. $\square$

4. Numerical results when $\kappa = \frac{3}{2}$

When $\kappa = \frac{3}{2}$, we have $\alpha_\kappa^D = 3.9114...$, $\beta_\kappa^D = 3.11582...$ [1]. In particular, we have $\alpha_\kappa^D < \beta_\kappa^D + 1$, so Corollary 1 can be applied to s in the range $\alpha_\kappa^D < s < \beta_\kappa^D + 1$ with $t = \frac{s}{s - \beta_\kappa^D}$. The improvement to the value of $F_\kappa(s)$ in this range is nonzero, but very small. Combining this with ordinary Buchstab iteration for the lower bound, one can show that $\beta(\frac{3}{2}) < 3.11570$.

If we apply the iteration from Corollary 2 directly to F_κ^D, f_κ^D, then the values of s, t for which the quantity $s^\kappa f_\kappa(s)$ is improved the most are given by $s \approx 4.85, t \approx 5.52$. This results in the bound $\beta(\frac{3}{2}) < 3.11554$.

Iteratively combining the improvements from Corollaries 1 and 2, we get $\beta(\frac{3}{2}) < 3.11549$.

5. An infinite sequence of iteration rules

Here we will describe an infinite sequence of iteration rules, one for each $k \ge 1$, generalizing the upper and lower bound iteration rules described so far (which correspond to the cases $k = 1$ and $k = 2$). We will also prove an optimality result for these iteration rules.

Theorem 5. *If $k \geq 1$ and $w \leq z^k$, then*

$$\begin{aligned}(-1)^{k-1}\mathcal{S}(A,z) \leq (-1)^{k-1}\mathcal{S}\big(A,w^{1/k}\big) &+ (-1)^{k-2}\sum_{w^{1/k}\leq p_1<z}\mathcal{S}\big(A_{p_1},(\tfrac{w}{p_1})^{1/(k-1)}\big)+\cdots\\
&+\sum_{(\frac{w}{p_1\cdots p_{k-2}})^{1/2}\leq p_{k-1}<\cdots<p_1<z}\mathcal{S}\big(A_{p_1\cdots p_{k-1}},\tfrac{w}{p_1\cdots p_{k-1}}\big)\\
&-\Big(1-\tfrac{1}{\binom{k+2}{2}}\Big)\sum_{\frac{w}{p_1\cdots p_{k-1}}\leq p_k<\cdots<p_1<z}\mathcal{S}\big(A_{p_1\cdots p_k},\tfrac{w}{p_1\cdots p_{k-1}}\big)\\
&+\sum_{\frac{w}{p_1\cdots p_{k-1}}\leq p_{k+1}<\cdots<p_1<z}\left(1-\frac{\#\{i\leq k+1 \mid wp_i\leq p_1\cdots p_{k+1}\}}{\binom{k+2}{2}}\right)\mathcal{S}\big(A_{p_1\cdots p_{k+1}},\tfrac{w}{p_1\cdots p_{k-1}}\big).\end{aligned}$$

Proof. It's enough to prove this when A has just one element, say $A = \{a\}$. We may also assume that a is squarefree, and write $a = q_1q_2\cdots q_m$ with $q_1 < q_2 < \cdots < q_m$ and the q_is prime. We may assume also that $q_1 < z$, since otherwise the result is trivial. Thus we just need to prove that the right hand side is at least 0.

Note that every nonzero summand corresponds to some divisor $d = p_1\cdots p_j$ of a having j prime factors, $j \leq k+1$. Our strategy is to combine the nonzero summands into small groups according to the combinatorial structure of their prime factors, such that each group of summands has a nonnegative sum.

The first step is to combine the summand corresponding to $d = p_1\cdots p_j$ with $j \leq k-1$ and $p_j = q_1$ with the summand corresponding to d/p_j, and to note that these two summands exactly cancel each other out. After this step, the only summands that remain are those which have $d = p_1\cdots p_j$ with $j \geq k-1$ and $p_{k-1} > q_1$.

The next step is to group the summands corresponding to $d = p_1\cdots p_j$ with $j \geq k-1$, $p_{k-1} = q_l$ with $l > 1$, and $p_1\cdots p_{k-1}$ taking some fixed value P with $w \leq Pq_1$. If $l = 2$, then the total contribution from such d is $\frac{1}{\binom{k+2}{2}}$. If $l = 3$, then the total contribution from such d is

$$1-\Big(1-\tfrac{1}{\binom{k+2}{2}}\Big)\cdot 2+\left(1-\frac{\#\{p\in\{p_1,\ldots,p_{k-1},q_2,q_1\}\mid wp\leq Pq_2q_1\}}{\binom{k+2}{2}}\right)=-\frac{\#\{i\leq k-1\mid wp_i\leq Pq_2q_1\}}{\binom{k+2}{2}}.$$

If $l = 4$, then the total contribution from such d is at least

$$1-\Big(1-\tfrac{1}{\binom{k+2}{2}}\Big)\cdot 3+\Big(1-\tfrac{k+1}{\binom{k+2}{2}}\Big)\cdot 3=\frac{\binom{k-1}{2}}{\binom{k+2}{2}}.$$

Finally, if $l \geq 5$ then the total contribution from such d is easily seen to be positive.

In order to balance out the negative contribution coming from groups corresponding to $P = p_1\cdots p_{k-1}, w \leq Pq_1, p_{k-1} = q_l$ with $l = 3$, we will assign portions of the positive excess from groups corresponding to Ps with $l = 2$ or $l = 4$ to certain corresponding Ps with $l = 3$.

If $P = p_1\cdots p_{k-1}, w \leq Pq_1, p_{k-1} = q_l$ with $l = 2$ and $m \geq 3$ is minimal such that q_m does not divide P, then we group the excess $\frac{1}{\binom{k+2}{2}}$ contribution from this P with the contributions corresponding to $P' = Pq_m/q_2$ - note that the least prime factor of P' is then necessarily equal to q_3.

If $P = p_1 \cdots p_{k-1}, w \le Pq_1, p_{k-1} = q_l$ with $l = 4$, then we take $\frac{\binom{k-1}{2}}{\binom{k+2}{2}}$ of the excess contribution from this P, and divide it into $k-2$ pieces of sizes $\frac{1}{\binom{k+2}{2}}, \frac{2}{\binom{k+2}{2}}, \ldots, \frac{k-2}{\binom{k+2}{2}}$, and we assign the piece of size $\frac{i}{\binom{k+2}{2}}$ to $P_i' = Pq_3/p_{i+1}$ (noting once again that P_i' has least prime factor equal to q_3).

To finish the argument, we just have to show that for $P = p_1 \cdots p_{k-1}, w \le Pq_1, p_{k-1} = q_l$ with $l = 3$, the total excess contribution that was assigned to P by the process described in the last two paragraphs is at least

$$\frac{\#\{i \le k-1 \mid wp_i \le Pq_2q_1\}}{\binom{k+2}{2}}.$$

To see this, let $m \ge 4$ be minimal such that q_m does not divide P (or let $m = k+2$ if $Pq_2q_1 = a$). For any $3 \le j < m$, if we let $P_j' = Pq_2/q_j$, then the least prime factor of P_j' is q_2, and as long as $wq_j \le Pq_2q_1$ we have $w \le P_j'q_1$ and the excess of $\frac{1}{\binom{k+2}{2}}$ corresponding to P_j' is assigned to P. Additionally (in the case $m < k+2$) we let $P' = Pq_m/q_3$, and we see that the least prime factor of P' is q_4, that $w \le Pq_1 < P'q_1$, and that $\frac{k+2-m}{\binom{k+2}{2}}$ of the excess corresponding to P' is assigned to P. Together, we see that the amount of excess which was assigned to P is at least

$$\frac{\#\{3 \le j < m \mid wq_j \le Pq_2q_1\}}{\binom{k+2}{2}} + \frac{k+2-m}{\binom{k+2}{2}} \ge \frac{\#\{i \le k-1 \mid wp_i \le Pq_2q_1\}}{\binom{k+2}{2}}. \qquad \square$$

To see that the kth iteration rule is optimal when we set $\kappa = 1$, $w = \frac{y}{z^2}$, and $y = z^s$ with $k + \frac{3}{2} < s < k+2$, we argue as in Theorem 4 to see that we just need to prove the following bound.

Theorem 6. *If $A^{\pm}$ are weighted sets of integers between 1 and y defined as in the discussion before Theorem 4, then for any $k \ge 1$, if $y = z^s$ with $k + \frac{3}{2} < s < k+2$ and $w = \frac{y}{z^2}$, we have*

$$\begin{aligned}
(-1)^{k-1}\mathcal{S}(A^{-^{k-1}}, z) = (-1)^{k-1}\mathcal{S}\big(A^{-^{k-1}}, w^{1/k}\big) + (-1)^{k-2} \sum_{w^{1/k} \le p_1 < z} \mathcal{S}\big(A_{p_1}^{-^{k-2}}, (\tfrac{w}{p_1})^{1/(k-1)}\big) + \cdots \\
+ \sum_{\left(\frac{w}{p_1 \cdots p_{k-2}}\right)^{1/2} \le p_{k-1} < \cdots < p_1 < z} \mathcal{S}\big(A^+_{p_1 \cdots p_{k-1}}, \tfrac{w}{p_1 \cdots p_{k-1}}\big) \\
- \Big(1 - \frac{1}{\binom{k+2}{2}}\Big) \sum_{\frac{w}{p_1 \cdots p_{k-1}} \le p_k < \cdots < p_1 < z} \mathcal{S}\big(A^-_{p_1 \cdots p_k}, \tfrac{w}{p_1 \cdots p_{k-1}}\big) \\
+ \sum_{\frac{w}{p_1 \cdots p_{k-1}} \le p_{k+1} < \cdots < p_1 < z} \left(1 - \frac{\#\{i \le k+1 \mid wp_i \le p_1 \cdots p_{k+1}\}}{\binom{k+2}{2}}\right) \mathcal{S}\big(A^+_{p_1 \cdots p_{k+1}}, \tfrac{w}{p_1 \cdots p_{k-1}}\big) \\
+ O\Big(\frac{y}{\log(z)^3}\Big).
\end{aligned}$$

Proof. Suppose that $a \le y$ is counted a different number of times on both sides of the above. Then we necessarily have $\lambda(a) = (-1)^k$, and the least prime dividing a is less than z. In order for the contribution of a to the right hand side to be positive, there must be primes $p_1 > \cdots > p_{k-1}$ dividing a such that $p_1 < z$ and such that the least prime dividing a is at least $\frac{w}{p_1 \cdots p_{k-1}}$, so we conclude that any prime dividing a must be at least

$$\frac{w}{p_1 \cdots p_{k-1}} > \frac{w}{z^{k-1}} = \frac{y}{z^{k+1}} = z^{s-(k+1)} > \sqrt{z}.$$

In particular, the number of such a which have a square factor is $O(\frac{y}{\sqrt{z}})$, so we may assume without loss that a is square free. If a has at least $k+4$ prime factors, then since a has some collection of

k prime factors whose product is at least w we have $a > w\sqrt{z}^4 = y$, a contradiction. Thus a has strictly less than $k+4$ prime factors, and since $\lambda(a) = (-1)^k$ we see that a has either k or $k+2$ prime factors.

If a has exactly k prime factors, then they must all be less than z in order for the contribution of a to the right hand side to be positive, so $a < z^k < \frac{y}{z^{3/2}}$, so the number of such a is at most $\frac{y}{z^{3/2}}$. Thus we may assume without loss that a has exactly $k+2$ prime factors, at least k of which are less than z.

If two of the prime factors of a are $\geq z$, then the remaining prime factors of a must have product at least w, so $a > wz^2 = y$, a contradiction. If one of the prime factors of a is $\geq z$ and the remaining $k+1$ prime factors of a are all $< z$, then the total contribution of a to the right hand side is precisely 0. Thus, we may assume that all of the prime factors of a are less than z.

If every product of k prime factors of a is $\geq w$, then the contribution of a is again precisely 0. Otherwise, we can write $a = q_1 \cdots q_{k+2}$ with $\sqrt{z} < q_1 < \cdots < q_{k+2}$, $q_1 \cdots q_k < w$, $q_{k+1} < z$, and $q_{k+2} < z$. Using an upper bound sieve to bound the number of possible values for $q_1 \cdots q_k$ by $O(\frac{w}{\log(z)})$, we see that the number of such a is $O(\frac{wz^2}{\log(z)^3}) = O(\frac{y}{\log(z)^3})$. $\square$

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