SYMMETRIC FEWNOMIAL INEQUALITIES AND HIGHER ORDER CONVEXITY

ZARATHUSTRA BRADY

Abstract. We prove a difficult symmetric fewnomial inequality by proving a more general inequality which applies to all functions with nonnegative fourth derivative.

1. Introduction

We start by introducing a convenient shorthand.

Definition 1.1. For $x, y, z > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$, let

$$[\alpha, \beta, \gamma] := \sum_{sym} x^\alpha y^\beta z^\gamma.$$

More generally, if f is any function and $x, y, z, \alpha, \beta, \gamma \in \mathbb{R}$, let

$$\{\alpha, \beta, \gamma\}_f := \sum_{sym} f(\alpha x + \beta y + \gamma z).$$

Note that the variables x, y, z are suppressed in this notation.

A special case of Muirhead's inequality states that if $(a, b, c), (p, q, r) \in \mathbb{R}^3$ satisfy $a \geq b \geq c$ and $p \geq q \geq r$, then the inequality

$$[a, b, c] \geq [p, q, r]$$

holds for all positive x, y, z if and only if we have

$$\begin{aligned} a &\geq p, \\ a + b &\geq p + q, \\ a + b + c &= p + q + r, \end{aligned}$$

and in this case we have the more general inequality

$$\{a, b, c\}_f \geq \{p, q, r\}_f$$

for every convex function f.

From the fewnomial point of view, this tells us that the differences $[a, b, c] - [p, q, r]$ should be viewed as having the same level of complexity as quadratic polynomials, despite the "fewnomial degree" (that is, the number of monomials minus 1) of such a difference being 11 in the general case. Thus it seems reasonable to conjecture that *symmetric* fewnomials, such as

$$\sum_{i=1}^{k} w_i [a_i, b_i, c_i],$$

for $a_i, b_i, c_i, w_i \in \mathbb{R}$, should have similar properties to polynomials of degree k.

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In particular, we may conjecture that symmetric fewnomial inequalities of the form

$$[a_1, b_1, c_1] + [a_2, b_2, c_2] \geq [a_3, b_3, c_3] + [a_4, b_4, c_4]$$

are true if they have equality to the fourth order at some point and if they are true in limiting cases, since the same is true for fourth degree polynomial inequalities in a single variable. To simplify things, we'll restrict to the case of homogeneous inequalities with total degree 0 which have fourth order equality at the point $(x, y, z) = (1, 1, 1)$.

In order to simplify things for ourselves further we will impose an additional symmetry: we will require our inequality to be invariant under replacing all variables with their inverses. It is not too hard to see that the most interesting such inequality should have the form

$$[a + b, 0, -a - b] + [c, 0, -c] \geq [a + b, -b, -a] + [a, b, -a - b],$$

with $a^2 + b^2 = c^2$ in order to make the second derivatives of the two sides match at $(x, y, z) = (1, 1, 1)$. Our main result is a proof of this inequality.

Theorem 1.1. *For $a, b, c \geq 0$ with $a^2 + b^2 = c^2$ we have*

$$[a + b, 0, -a - b] + [c, 0, -c] \geq [a + b, -b, -a] + [a, b, -a - b].$$

More generally, for f any function with nonnegative fourth derivative we have

$$\{a + b, 0, -a - b\}_f + \{c, 0, -c\}_f \geq \{a + b, -b, -a\}_f + \{a, b, -a - b\}_f.$$

2. Reduction

We wish to show that for x, y, z arbitrary, $a^2 + b^2 = c^2$, $a, b, c \geq 0$, and f a function with $f^{(4)} \geq 0$, we have

$$\{a + b, 0, -a - b\}_f + \{c, 0, -c\}_f \geq \{a + b, -b, -a\}_f + \{a, b, -a - b\}_f.$$

Because of the sign symmetry, we can assume that $f(x) = f(-x)$, by replacing $f(x)$ by $f(x) + f(-x)$ if necessary.

Note that by rescaling and rearranging, we can assume without loss of generality that $b = 1$ and $a \geq 1$. Next, if we replace x, y, z with $y-z, z-x, x-y$, then the above becomes

$$\{a + 1, 0, 0\}_f + \{\sqrt{a^2 + 1}, 0, 0\}_f \geq \{-a, 1, 0\}_f + \{a, -1, 0\}_f,$$

with the restriction that $x + y + z = 0$ and that $f(x) = f(-x)$. By changing signs and rearranging we assume that $x \geq y \geq 0 \geq z$. By continuity we can assume that $y > 0$, and then by rescaling that $y = 1$, so $z = -x - 1$.

Thus, if we set

$$\begin{aligned} D_f(a, x) :=& f((a + 1)(x + 1)) + f(c(x + 1)) + f((a + 1)x) \\ &+ f(cx) + f(a + 1) + f(c) \\ &- f(ax + x + a) - f(ax + x + 1) - f(ax + a + 1) \\ &- f(ax - 1) - f(x + a + 1) - f(x - a), \end{aligned}$$

where $c = \sqrt{a^2 + 1}$, then we need to show that $D_f(a, x) \geq 0$ for $a, x \geq 1$.

We can assume without loss of generality that f is of the form $f(x) = (x-t)_+^3 + (-x-t)_+^3$ for some $t \geq 0$, since every even function with nonnegative fourth derivative can be written as a limit of linear combinations of such functions plus a quadratic.

Set

$$r_k(a, x, t) = D_{f_{k,t}}(a, x),$$

where $f_{k,t}(x) = (x - t)_+^k + (-x - t)_+^k$. Then we just need to show that $r_3(a, x, t) \geq 0$ for $a, x \geq 1$, $t \geq 0$.

Note that $\frac{\partial}{\partial t} r_k(a,x,t) = -k r_{k-1}(a,x,t)$, and that for $k = 0,1,2,3$ we have $r_k(a,x,\infty) = 0$.

3. Argument

Throughout this section we will maintain the following notation. The letters a, x will always denote real numbers ≥ 1, the letter c will always mean $\sqrt{a^2+1}$, and for $t \geq 0$ and $k \in \mathbb{N}$ the expression $r_k(a,x,t)$ will always be given by

$$\begin{aligned} r_k(a,x,t) = {} & ((a+1)(x+1)-t)_+^k + (c(x+1)-t)_+^k + ((a+1)x-t)_+^k \\ & + (cx-t)_+^k + (a+1-t)_+^k + (c-t)_+^k - (ax+x+a-t)_+^k \\ & - (ax+x+1-t)_+^k - (ax+a+1-t)_+^k - (ax-1-t)_+^k \\ & - (x+a+1-t)_+^k - (x-a-t)_+^k - (a-x-t)_+^k. \end{aligned}$$

Our goal is to prove that $r_3(a,x,t) \geq 0$ for $a, x \geq 1$ and $t \geq 0$. The general approach is to use the fact that $\frac{\partial^2}{\partial t^2} r_3(a,x,t) = 6 r_1(a,x,t)$ to determine where $r_3(a,x,t)$ is convex or concave (in t), taking full advantage of the fact that $r_1(a,x,t)$ is piecewise linear in both x and t.

Lemma 3.1. *For $a, x \geq 1$ we always have*

$$r_3(a,x,0) \geq 0.$$

If we assume furthermore that

$$x \leq \frac{c}{a+1-c},$$

then we also have

$$r_1(a,x,0) \geq 0.$$

Proof. We have

$$\begin{aligned} r_3(a,x,0) &= (a+1)^3(x+1)^3 + c^3(x+1)^3 + (a+1)^3x^3 + c^3x^3 + (a+1)^3 + c^3 - ((a+1)x+a)^3 \\ &\quad - ((a+1)x+1)^3 - (ax+a+1)^3 - (ax-1)^3 - (x+a+1)^3 - |x-a|^3 \\ &= (x+1)((c^3-a^3)(2x^2+x+2) - (6a+3)x) - 2(x-a)_+^3. \end{aligned}$$

Noting that $c^3 \geq a^3 + \frac{6}{5}a + \frac{3}{5}$, which follows from

$$(a^2+1)^3 - \left(a^3 + \frac{6}{5}a + \frac{3}{5}\right)^2 = \frac{3}{5}a^2(a-1)^2 + \frac{16}{25}(a-1)^2 + \frac{8}{25}a^2 - \frac{4}{25}a \geq 0,$$

we see that

$$r_3(a,x,0) \geq \frac{6}{5}(2a+1)(x+1)(x-1)^2 - 2(x-a)_+^3 \geq 0.$$

For the second claim, we have

$$\begin{aligned} r_1(a,x,0) &= (a+1)(x+1) + c(x+1) + (a+1)x + cx + (a+1) + c - ((a+1)x+a) \\ &\quad - ((a+1)x+1) - (ax+a+1) - (ax-1) - (x+a+1) - |x-a| \\ &= 2(x+1)c - (2ax+x+a+|x-a|). \end{aligned}$$

If $x \leq a$, this becomes

$$r_1(a,x,0) = 2((x+1)c - (x+1)a) \geq 0.$$

Finally, if $a \leq x \leq \frac{c}{a+1-c}$, then

$$r_1(a,x,0) = 2(c-(a+1-c)x) \geq 0. \qquad \square$$

Lemma 3.2. *If $a \geq \sqrt{3}$, then we have*

$$\frac{2a+2-c}{2c-a-1} \leq \frac{2c-a-1}{2a+1-2c} \leq \frac{c}{a+1-c} \leq a+1 \leq c+a-1,$$

and we have equality when $a = \sqrt{3}$. When $1 \leq a \leq \sqrt{3}$, the outer inequality is reversed.

Proof. For the first inequality, we have

$$(2c-a-1)^2 - (2a+2-c)(2a+1-2c) = (c-2)(2a-c-1) \geq 0.$$

For the second inequality, we have

$$c(2a+1-2c) - (2c-a-1)(a+1-c) = (c-2)(c-a) \geq 0.$$

The third inequality follows from

$$(a+1)(a+1-c) - c = (c-2)(c-a) \geq 0,$$

and the last inequality is immediate.

Finally, when $1 \leq a \leq \sqrt{3}$, we have

$$(2a+2-c) - (c+a-1)(2c-a-1) = (2-c)(c+a) \geq 0. \qquad \square$$

Lemma 3.3. *If we assume that*

$$\frac{2a+2-c}{2c-a-1} \leq x \leq c+a-1,$$

then

$$r_2(a,x,cx) \geq x.$$

Proof. Note that in this case we have $a \geq \sqrt{3}$ and

$$(c-1)x \geq (c-1)\frac{2a+2-c}{2c-a-1} = a+1+\frac{c-2}{2c-a-1} \geq a+1,$$

and

$$(c-a)x \leq 1+a-c < a+1.$$

Thus,

$$\begin{aligned}
r_2(a,x,cx) &= ((a+1)(x+1)-cx)_+^2 + (c(x+1)-cx)_+^2 + ((a+1)x-cx)_+^2 \\
&\quad + (cx-cx)_+^2 + (a+1-cx)_+^2 + (c-cx)_+^2 - ((a+1)x+a-cx)_+^2 \\
&\quad - ((a+1)x+1-cx)_+^2 - (ax+a+1-cx)_+^2 - (ax-1-cx)_+^2 \\
&\quad - (x+a+1-cx)_+^2 - (x-a-cx)_+^2 - (a-x-cx)_+^2 \\
&= ((a+1-c)x+a+1)^2 + c^2 + (a+1-c)^2x^2 - ((a+1-c)x+a)^2 \\
&\quad - ((a+1-c)x+1)^2 - (a+1-(c-a)x)^2 \\
&= (-(c-a)x+2(a+1))(c-a)x \\
&\geq (c+a+1)(c-a)x \\
&\geq x. \qquad \square
\end{aligned}$$

Lemma 3.4. *If $a, x \geq 1$, then we have*

$$\begin{aligned}
r_1(a, x, (a+1)x + a + 1) &= 0,\\
r_1(a, x, (a+1)x + a) &= 1,\\
r_1(a, x, (a+1)x + 1) &> 0,\\
r_1(a, x, ax + a + 1) &\geq 0,\\
r_1(a, x, a + 1) &< 0,\\
r_1(a, x, c) &\leq 0.
\end{aligned}$$

Proof. The first two are immediate. For the third, note

$$\begin{aligned}
r_1(a, x, (a+1)x + 1) &= a + (-(a+1-c)x + c - 1)_+ - (a-1) - (a-x)_+\\
&= 1 + (-(a+1-c)x + c - 1)_+ - (a-x)_+.
\end{aligned}$$

If $x > a - 1$, then this is obviously positive. If $x \leq a - 1$, then

$$\begin{aligned}
r_1(a, x, (a+1)x + 1) &= 1 + (-(a+1-c)x + c - 1) - (a-x)\\
&= (c-a)(x+1) > 0.
\end{aligned}$$

For the fourth, we have

$$\begin{aligned}
r_1(a, x, ax + a + 1) &= x + ((c-a)x - (a+1-c))_+ + (x - (a+1))_+\\
&\quad + ((c-a)x - (a+1))_+ - (x-1) - (x-a)_+\\
&\geq 1 + (x - (a+1))_+ - (x-a)_+ \geq 0.
\end{aligned}$$

For the fifth,

$$\begin{aligned}
r_1(a, x, a+1) &= (a+1)x + (cx - (a+1-c)) + ((a+1)x - (a+1)) + (cx - (a+1))_+\\
&\quad - ((a+1)x - 1) - ((a+1)x - a) - ax - (ax - (a+2))_+ - x - (x - (2a+1))_+\\
&= \; -(a+1-c)(x+1) + (cx - (a+1))_+ - (ax - (a+2))_+ - (x - (2a+1))_+.
\end{aligned}$$

This is maximized when x is either 1 or $\frac{a+2}{a}$. If $x = \frac{a+2}{a}$, then it becomes

$$-(a+1-c)\frac{2a+2}{a} + \left(c\frac{a+2}{a} - (a+1)\right) = -\frac{1}{a}\left(3a^2 + 5a + 2 - (3a+4)c\right) < 0,$$

where the last inequality follows from

$$(3a^2 + 5a + 2)^2 - (3a+4)^2(a^2+1) = 6a^3 + 12a^2 - 4a - 12 > 0.$$

Finally,

$$\begin{aligned}
r_1(a, x, c) &= ((a+1)x + a + 1 - c) + cx + ((a+1)x - c) + (cx - c) + (a + 1 - c)\\
&\quad - ((a+1)x + a - c) - ((a+1)x + 1 - c) - (ax + a + 1 - c) - (ax - (c+1))_+\\
&\quad - (x + a + 1 - c) - (x - (a+c))_+\\
&= (2c - a - 1)x - (a+1) - (ax - (c+1))_+ - (x - (a+c))_+.
\end{aligned}$$

This is maximized when $x = \frac{c+1}{a}$, so it's at most

$$(2c - a - 1)\frac{c+1}{a} - (a+1) = -\frac{1}{a}(c - a + 1)(a - 1) \leq 0. \qquad \square$$

Lemma 3.5. *For $a, x \geq 1$, we have*

$$r_1(a, x, cx + c) < 0$$

if and only if

$$\frac{2c - a - 1}{2a + 1 - 2c} < x < c + a - 1.$$

Proof.

$$\begin{aligned} r_1(a, x, cx + c) &= ((a + 1 - c)x + a + 1 - c) + ((a + 1 - c)x - c)_+ - ((a + 1 - c)x - (c - a)) \\ &\quad - ((a + 1 - c)x - (c - 1))_+ - (-(c - a)x + a + 1 - c)_+ - (-(c - 1)x + a + 1 - c)_+ \\ &= 1 + ((a + 1 - c)x - c)_+ - ((a + 1 - c)x - (c - 1))_+ \\ &\quad - (-(c - a)x + a + 1 - c)_+ - (-(c - 1)x + a + 1 - c)_+. \end{aligned}$$

This is certainly nonnegative if $(c-a)x \geq a+1-c$, or equivalently if $x \geq c+a-1$, since then the last two terms are 0. If $\frac{c}{a+1-c} < x < c + a - 1$, then all terms except possibly the last will be nonzero and $r_1(a, x, cx + c)$ will be negative.

Note that

$$3c - 2(a + 1) = \frac{c^2 + 4(a - 1)^2}{3c + 2(a + 1)} > 0.$$

Thus, when $(a+1-c)x \leq (c-1)$ or $(c-1)x \leq (a+1-c)$, $r_1(a, x, cx+c)$ is a nondecreasing function of x, and when $x = 1$,

$$r_1(a, 1, 2c) = 2(c - a) - 2(a + 2 - 2c)_+ > 0.$$

When $a > \sqrt{3}$, both $\frac{a+1-c}{c-1}$ and $\frac{c-1}{a+1-c}$ are less than $\frac{2c-a-1}{2a+1-2c}$.

Thus, the only remaining case is the case where $(c-1) < (a+1-c)x$, $(a+1-c) < (c-1)x$, $x < c + a - 1$, $x \leq \frac{c}{a+1-c}$. In this case,

$$r_1(a, x, cx+c) = 1-((a+1-c)x-(c-1))-(-(c-a)x+a+1-c) = 2c-a-1-(2a+1-2c)x,$$

which is less than zero if and only if

$$x > \frac{2c - a - 1}{2a + 1 - 2c}. \qquad \square$$

Lemma 3.6. *If $a, x \geq 1$, then $r_1(a, x, x + a + 1) < 0$ if and only if*

$$\frac{c}{2a - c} < x < \min\left(\frac{2a + 2 - c}{2c - a - 1}, c + a - 1\right).$$

Proof.

$$\begin{aligned} r_1(a, x, x + a + 1) &= (ax) + ((c - 1)x - (a + 1 - c))_+ + (ax - (a + 1))_+ + ((c - 1)x - (a + 1))_+ \\ &\quad - (ax - 1) - (ax - a) - ((a - 1)x) - ((a - 1)x - (a + 2))_+ \\ &= -(2a - 1)x + a + 1 + ((c - 1)x - (a + 1 - c))_+ + (ax - (a + 1))_+ \\ &\quad + ((c - 1)x - (a + 1))_+ - ((a - 1)x - (a + 2))_+. \end{aligned}$$

Note that for $x \leq \frac{c}{a-c}$, we have $(c - 1)x < a + 1$, since

$$(2a - c)(a + 1) - c(c - 1) = (c - a)c + 2(a - 1) > 0,$$

so we have

$$\begin{aligned} r_1(a, x, x + a + 1) &= -(2a - 1)x + a + 1 + ((c - 1)x - (a + 1 - c))_+ + (ax - (a + 1))_+ \\ &\geq -(2a - c)x + c \geq 0. \end{aligned}$$

Note that if $\frac{c}{2a-c} < c+a-1$, we have $a\frac{c}{2a-c} < a+1$:

$$\begin{aligned}(2a-c)(a+1) - ac = 2a^2 - 2ac + 2a - c &= \frac{4a^3 - a^2 - 4a - 1}{2a^2 + 2a + 2ac + c} \\ &= (a^2 - 2a - 1 + ac)\frac{ac - a^2 + 2a + 1}{2a^2 + 2a + 2ac + c} \\ &= ((2a-c)(c+a-1) - c)\frac{ac - a^2 + 2a + 1}{2a^2 + 2a + 2ac + c} > 0.\end{aligned}$$

Furthermore, if $x \geq \frac{c}{2a-c}$, then $(c-1)x > a+1-c$:

$$c(c-1) - (2a-c)(a+1-c) = 3ac - 2a^2 - 2a + 2 \geq (a-1)^2 + 1 > 0.$$

Thus, if $\frac{c}{2a-c} < c+a-1$ we see that $r_1(a, x, x+a+1)$ is zero when $x = \frac{c}{2a-c}$, and is less than zero for $\frac{c}{2a-c} < x \leq 1 + \frac{1}{a}$. For $x \geq 1 + \frac{1}{a}$, $r_1(a, x, x+a+1)$ becomes an increasing function of x, so we just need to check that it is equal to zero when $x = \min\left(\frac{2a+2-c}{2c-a-1}, c+a-1\right)$.

When $x \geq c+a-1$, we have $(c-a)x \geq a+1-c$, so

$$\begin{aligned}r_1(a, x, x+a+1) = &-(2a-c)x + c + (ax - (a+1))_+ \\ &+ ((c-1)x - (a+1))_+ - ((a-1)x - (a+2))_+ \\ \geq &\ (c-a)x - (a+1-c) \geq 0,\end{aligned}$$

with equality when $x = c+a-1$, $a(c+a-1) \geq a+1$, $(c-1)(c+a-1) \leq a+1$. Note that we have $a(c+a-1) \geq a+1$ if and only if $c+a-1 \geq \frac{c}{2a-c}$, since $(2a-c)(c+a-1) - c = a(c+a-1) - (a+1)$. We also have $(c-1)(c+a-1) \leq a+1$ if and only if $a \leq \sqrt{3}$:

$$a + 1 - (c-1)(c+a-1) = 2a - a^2 - 1 - ac + 2c = (a+c)(2-c).$$

Finally, assume $a \geq \sqrt{3}$ and $x \geq \frac{2a+2-c}{2c-a-1}$. Recall that in this case we have $(c-1)x \geq a+1$, so

$$r_1(a, x, x+a+1) = (2c-a-1)x - (2a+2-c) - ((a-1)x - (a+2))_+.$$

Since this is an increasing function of x, we just have to check that equality holds when $x = \frac{2a+2-c}{2c-a-1}$. Since

$$(a+2)(2c-a-1) - (a-1)(2a+2-c) = 3(c-a)(a+1) > 0,$$

we have

$$r_1(a, \frac{2a+2-c}{2c-a-1}, \frac{2a+2-c}{2c-a-1} + a + 1) = 0. \qquad \square$$

Lemma 3.7. *If $x, a \geq 1$, then we have $r_1(a, x, |x-a|) < 0$ if and only if*

$$\frac{c}{a+1-c} < x < \frac{(1+2a)(a+c)-1}{2}.$$

Proof.

$$\begin{aligned}r_1(a, x, |x-a|) = &\ ((a+1)x + a + 1 - |x-a|) + (cx + c - |x-a|) + ((a+1)x - |x-a|) \\ &+ (cx - |x-a|) + (a+1-|x-a|)_+ + (c - |x-a|)_+ \\ &- ((a+1)x + a - |x-a|) - ((a+1)x + 1 - |x-a|) - (ax + a + 1 - |x-a|) \\ &- (ax - 1 - |x-a|) - (x + a + 1 - |x-a|) \\ = &-(1 - 2(c-a))x - (2a+1-c) + |x-a| + (a+1-|x-a|)_+ + (c - |x-a|)_+.\end{aligned}$$

Note that if $x \leq a$, then this becomes

$$r_1(a, x, a-x) = 2(c-a)x + 2(c-a),$$

which is clearly positive. If $x \geq a$, it becomes

$$r_1(a, x, x-a) = 2(c-a)x - (3a+1-c) + (2a+1-x)_+ + (a+c-x)_+.$$

If we now assume that $x \leq \frac{c}{a+1-c}$, then $x \leq a+1$, so

$$r_1(a, x, x-a) = -2(a+1-c)x + 2c \geq 0,$$

and we have equality when $x = \frac{c}{a+1-c}$. For $a \leq x \leq 2a+1$, $r_1(a, x, x-a)$ is a nonincreasing function of x (since $2(c-a) < 1$), and for $x \geq 2a+1$ we have

$$r_1(a, x, x-a) = 2(c-a)x - (3a+1-c),$$

which reaches 0 when

$$x = \frac{3a+1-c}{2(c-a)} = \frac{(1+2a)(a+c)-1}{2}. \qquad \square$$

Lemma 3.8. *If $x, a \geq 1$ and $ax - 1 > 1 + a$, then $r_1(a, x, ax-1)$ is negative if and only if*

$$x < \frac{2a+2-c}{2c-a-1}.$$

Proof. In this case,

$$\begin{aligned} r_1(a, x, ax-1) &= (x+a+2) + ((c-a)x+c+1) + (x+1) \\ &\quad + ((c-a)x+1) - (x+a+1) - (x+2) \\ &\quad - (a+2) - (-(a-1)x+a+2)_+ \\ &= 2(c-a)x + c - a - (-(a-1)x+a+2)_+. \end{aligned}$$

This is certainly positive if $(a-1)x > a+2$. We have $\frac{2a+2-c}{2c-a-1} < \frac{a+2}{a-1}$, since

$$(2c-a-1)(a+2) - (2a+2-c)(a-1) = 3(c-a)(a+1) > 0.$$

If $(a-1)x \leq a+2$, we have

$$r_1(a, x, ax-1) = (2c-a-1)x - (2a+2-c),$$

which is negative exactly when

$$x < \frac{2a+2-c}{2c-a-1}. \qquad \square$$

Lemma 3.9. *Suppose $x, a \geq 1$. If $a < \sqrt{3}$, then we have $r_1(a, x, (a+1)x) < 0$ if and only if*

$$x < \max(c+a-1, \frac{2a+2-c}{c}),$$

and for $x \geq \frac{c}{a+1-c}$ we have $r_1(a, x, (a+1)x) = 0$. If $a \geq \sqrt{3}$, then $r_1(a, x, (a+1)x) < 0$ if and only if $x < a+1$, and for $x \geq a+1$ we have $r_1(a, x, (a+1)x) = 0$.

Proof.

$$\begin{aligned} r_1(a, x, (a+1)x) &= a + 1 + (-(a+1-c)x+c)_+ - a - 1 - (-x+a+1)_+ - (-ax+a+1)_+ \\ &= (-(a+1-c)x+c)_+ - (-x+a+1)_+ - (-ax+a+1)_+. \end{aligned}$$

The first term is 0 when $x \geq \frac{c}{a+1-c}$. When $a < \sqrt{3}$ and $x \leq c+a-1$, we have

$$r_1(a, x, (a+1)x) \leq (-(a+1-c)x+c) - (-x+a+1) = (c-a)x - (a+1-c) \leq (c-a)(c+a-1) - (a+1-c) = 0,$$

with equality when $x = c+a-1$ and $a(c+a-1) \geq a+1$. Note that we have $a(c+a-1) \geq a+1$ exactly when we have $c(c+a-1) \geq 2a+2-c$. Otherwise, for $ax < a+1$ we have

$$r_1(a,x,(a+1)x) = cx - (2a+2-c),$$

which is zero when $x = \frac{2a+2-c}{c}$, which is necessarily less than $\frac{a+1}{a}$.

When $a \geq \sqrt{3}$, note that for $x < a+1$, the derivative of $r_1(a,x,(a+1)x)$ with respect to x is at least $c-a$, and for $x \geq a+1$ we have $r_1(a,x,(a+1)x) = 0$. □

Lemma 3.10. *If $x, a \geq 1$ and $cx \geq 1+a$, then $r_1(a,x,cx) < 0$ if and only if*

$$x < \max\left\{\frac{2a+2-c}{2c-a-1}, c+a-1\right\}.$$

If $\frac{2a+2-c}{2c-a-1} < x < c+a-1$, then $r_1(a,x,cx) > -1$.

Proof. In this case, we have

$$\begin{aligned}
r_1(a,x,cx) &= ((a+1-c)x+a+1) + c + (a+1-c)x - ((a+1-c)x+a) \\
&\quad - ((a+1-c)x+1) - (-(c-a)x+a+1)_+ - (-(c-1)x+a+1)_+ \\
&= c - (-(c-a)x+a+1)_+ - (-(c-1)x+a+1)_+.
\end{aligned}$$

Note that this is an increasing function of x. When $a \geq \sqrt{3}$, $x = c+a-1$, we have $-(c-1)(c+a-1)+a+1 = -(c-2)(c+a) \leq 0$, so

$$r_1(a,x,c(c+a-1)) = c - (-(c-a)(c+a-1)+a+1) = c-c = 0.$$

When $a < \sqrt{3}$, we have $\frac{2a+2-c}{2c-a-1} < \frac{a+1}{c-1}$, since

$$(2c-a-1)(a+1) - (2a+2-c)(c-1) = 2-c > 0.$$

Thus, we have

$$\begin{aligned}
r_1(a,x,c\frac{2a+2-c}{2c-a-1}) &= c - \left(-(c-a)\frac{2a+2-c}{2c-a-1} + a+1\right) - \left(-(c-1)\frac{2a+2-c}{2c-a-1} + a+1\right) \\
&= -(2a+2-c) + (2c-a-1)\frac{2a+2-c}{2c-a-1} = 0.
\end{aligned}$$

Finally, when $\frac{2a+2-c}{2c-a-1} < x < c+a-1$, we have $(c-1)x \geq a+1$ and $(c-a)x \leq a+1$, so

$$r_1(a,x,cx) = c - (-(c-a)x+a+1) = (c-a)(x+1)-1 > -1.$$ □

Lemma 3.11. *There is no $0 \leq t_1 \leq t_2 \leq a+1$ such that $r_1(a,x,t_1) < 0$, $r_1(a,x,t_2) > 0$.*

Proof. Assume that such t_1 and t_2 exist for the sake of contradiction. For $0 \leq t \leq a+1$, we have

$$\begin{aligned}
r_1(a,x,t) &= ((a+1)x+a+1-t) + (cx+c-t) + ((a+1)x-t) \\
&\quad + (cx-t)_+ + (a+1-t) + (c-t)_+ - ((a+1)x+a-t) \\
&\quad - ((a+1)x+1-t) - (ax+a+1-t) - (ax-1-t)_+ \\
&\quad - (x+a+1-t) - (|x-a|-t)_+ \\
&= -(a+1-c)(x+1) + (cx-t)_+ + (c-t)_+ \\
&\quad - (ax-1-t)_+ - (|x-a|-t)_+.
\end{aligned}$$

Note that this is a nonincreasing function of t when $0 \leq t \leq c$. Thus we must have $c < t_2 \leq a+1$. Recall from Lemma 3.4 that $r_1(a,x,c) \leq 0$ and $r_1(a,x,a+1) < 0$. The only interval in which $r_1(a,x,t)$ can be an increasing function of t is the interval $c \leq t \leq x-a$, if it exists. Thus we may as well assume that $t_2 = x - a > c$, and $r_1(a, x, x - a) > 0$. In this case, note that in order for us to have $r_1(a, x, a + 1) < 0$, $r_1(a, x, t)$ must decrease somewhere between $x - a$ and $a + 1$, so we must have $ax - 1 < a + 1$. But then we would have

$$x > a + c = \frac{2a^2 + 2ac}{2a} \geq \frac{a^2 + ac + 1 + c}{2a} = \frac{c}{a + 1 - c}$$

and

$$x < 1 + \frac{2}{a} < 3 < \frac{3(1 + \sqrt{2}) - 1}{2} \leq \frac{(1 + 2a)(a + c) - 1}{2},$$

so by Lemma 3.7 we have $r_1(a, x, x - a) < 0$, a contradiction. □

Lemma 3.12. *If $|x - a| > a + 1$ and $r_1(a, x, |x - a|) > 0$, then $ax - 1 > a + 1$ and $r_1(a, x, ax - 1) > 0$, and $r_1(a, x, x + a + 1) \geq 0$ as well.*

Proof. Note that in this case, $x > 2a + 1 > \frac{(1+a+c)c}{2a} = \frac{c}{a+1-c}$, so by Lemma 3.7, we must have $x > \frac{(1+2a)(a+c)-1}{2} \geq \frac{2+3\sqrt{2}}{2}$. Since $ax - 1 \geq x - a > a + 1$, by Lemma 3.8 and Lemma 3.6 it is enough to show that $\frac{2+3\sqrt{2}}{2} \geq \frac{2a+2-c}{2c-a-1}$, which follows from

$$(2+3\sqrt{2})(2c-a-1)-2(2a+2-c) = 3(2+\sqrt{2})(\sqrt{2}c-a-1) = 3(2+\sqrt{2})\frac{(a-1)^2}{\sqrt{2}c+a+1} \geq 0.$$

□

Lemma 3.13. *If $r_1(a, x, cx + c) > 0$ and $t \geq cx + c$, then $r_1(a, x, t) \geq 0$.*

Proof. Since $r_1(a, x, t)$ is piecewise linear in t, we just have to check this for t equal to one of $(a + 1)x + a + 1$, $(a + 1)x + a$, $(a + 1)x + 1$, $ax + a + 1$, $(a + 1)x$, $x + a + 1$. The first four cases follow from Lemma 3.4.

If $t = (a + 1)x$, then from $t \geq cx + c$ we see that $x \geq \frac{c}{a+1-c}$. If $a \leq \sqrt{3}$, then by Lemma 3.9 we have $r_1(a, x, (a + 1)x) = 0$. If $a > \sqrt{3}$ then by Lemma 3.5 we must have $x > c + a - 1 \geq a + 1$, so by Lemma 3.9 we again have $r_1(a, x, (a + 1)x) = 0$.

Finally, if $t = x + a + 1$, then we must have $x \leq \frac{a+1-c}{c-1} = \frac{c+1-a}{a}$. By Lemma 3.6, it's enough to check that $\frac{c+1-a}{a} > \frac{c}{2a-c}$, and this follows from

$$ac - (c + 1 - a)(2a - c) = (c + 1 - a)(c - a) + a(a - 1) > 0.$$ □

Lemma 3.14. *If $r_1(a, x, (a + 1)x) > 0$ and $t \geq (a + 1)x$, then $r_1(a, x, t) \geq 0$.*

Proof. Similarly to the previous Lemma, we see that it is enough to check this when t is either $cx + c$ and $x + a + 1$. Note that if $r_1(a, x, (a + 1)x) > 0$, then we must have $a < \sqrt{3}$ by Lemma 3.9. Thus if $t = cx + c$, then $r_1(a, x, cx + c) \geq 0$ by Lemma 3.5. If $t = x + a + 1$, then by Lemma 3.9, we must have $x > c + a - 1$, and by Lemma 3.6 this implies that $r_1(a, x, x + a + 1) \geq 0$. □

Lemma 3.15. *If $cx \geq a + 1$, $r_1(a, x, cx) > 0$, and $t \geq cx$, then $r_1(a, x, t) \geq 0$.*

Proof. We just have to check this for t equal to one of $cx + c$, $x + a + 1$, $(a + 1)x$. Note that by Lemma 3.10 we have $x > \max\left(\frac{2a+2-c}{2c-a-1}, c + a - 1\right) \geq c + a - 1$, so by Lemma 3.5 we have $r_1(a, x, cx + c) \geq 0$ and by Lemma 3.6 we have $r_1(a, x, x + a + 1) \geq 0$. Finally, since

$$\max\left(\frac{2a + 2 - c}{2c - a - 1}, c + a - 1\right) \geq \max\left(\frac{c}{a + 1 - c}, a + 1\right),$$

we can apply Lemma 3.9 to see that we have $r_1(a, x, (a + 1)x) = 0$. □

Lemma 3.16. *Suppose $r_1(a,x,x+a+1) \geq 0$ and $t \geq x+a+1$. If $r_1(a,x,t) < 0$, then $t > ax-1$ and*

$$\frac{2a+2-c}{2c-a-1} \leq x \leq c+a-1.$$

Furthermore, in this case we also have $cx \geq x+a+1$.

Proof. First we check that we have $r_1(a,x,ax-1) \geq 0$ when $ax-1 \geq x+a+1$, that is, when $x \geq \frac{a+2}{a-1}$. Since $\frac{a+2}{a-1} > \frac{2a+2-c}{2c-a-1}$ (by the proof of Lemma 3.6), we can apply Lemma 3.8 to conclude that $r_1(a,x,ax-1) \geq 0$.

Now we consider the cases $t = cx, cx+c, (a+1)x$. If $t = cx$, then by Lemma 3.10 we have $x < c+a-1$, and from $cx \geq x+a+1$ we see that $x \geq \frac{a+1}{c-1} \geq \frac{c}{2a-c}$, so by Lemma 3.6 we have $x \geq \frac{2a+2-c}{2c-a-1}$. If $t = cx+c$, then Lemma 3.5 finishes immediately.

The final case is $t = (a+1)x$. In this case we must have $x \geq \frac{a+1}{a}$, and by the proof of Lemma 3.6 this is at least $\min\left(\frac{c}{2a-c}, c+a-1\right)$, so we must have $x \geq \min\left(\frac{2a+2-c}{2c-a-1}, c+a-1\right)$. For $a \geq \sqrt{3}$, we conclude from Lemma 3.9 that $\frac{2a+2-c}{2c-a-1} \leq x \leq a+1 \leq c+a-1$. For $a < \sqrt{3}$, we have $x \geq c+a-1$, so we can apply Lemma 3.9 to see that $x < \frac{2a+2-c}{c}$. But this is impossible: if $\frac{2a+2-c}{c} > x \geq c+a-1$, then we have $x \geq \frac{a+1}{a} > \frac{2a+2-c}{c}$, since

$$\begin{aligned}(a+1)c - a(2a+2-c) = 2ac + c - 2a^2 - 2a &= \frac{1+4a+a^2-4a^3}{2ac+c+2a^2+2a} \\ &= (2a+1-a^2-ac)\frac{ac-a^2+2a+1}{2ac+c+2a^2+2a} \\ &= ((2a+2-c) - c(c+a-1))\frac{ac-a^2+2a+1}{2ac+c+2a^2+2a} > 0.\end{aligned}$$

To show that $cx \geq x+a+1$, we note that by Lemma 3.2, $\frac{2a+2-c}{2c-a-1} \leq c+a-1$ implies that $a \geq \sqrt{3}$, and so by the proof of Lemma 3.3 we have $(c-1)x \geq a+1$. $\square$

Lemma 3.17. *Suppose that $ax-1 \geq a+1$, $r_1(a,x,ax-1) \geq 0$, and $t \geq ax-1$. If $r_1(a,x,t) < 0$, then we must have $t > x+a+1$ and*

$$\frac{2a+2-c}{2c-a-1} \leq x \leq c+a-1.$$

Proof. Note that by Lemma 3.8, we have $x \geq \frac{2a+2-c}{2c-a-1}$. By Lemma 3.6, it follows that $r_1(a,x,x+a+1) \geq 0$. It remains to show that if t is any of $cx, cx+c$, or $(a+1)x$, then $t > x+a+1$ and $x \leq c+a-1$. Note that we always have $cx > ax-1$ and $cx+c, (a+1)x > cx$, so in fact we only have to check this for $t = cx$.

When $t = cx$, then Lemma 3.10 yields $x < c+a-1$. Thus $a \geq \sqrt{3}$, so we have $cx > (x+a+1)$, since $(c-1)\frac{2a+2-c}{2c-a-1} \geq a+1$ (by the proof of Lemma 3.3). $\square$

Lemma 3.18. *If t_1, t_2 are such that $a+1 \leq t_1 \leq t_2$ and $r_1(a,x,t_1) > 0$, $r_1(a,x,t_2) < 0$, then $t_1 < cx$, $t_2 > \max(x+a+1, ax-1)$, and*

$$\frac{2a+2-c}{2c-a-1} \leq x \leq c+a-1.$$

Proof. First we assume for contradiction that $t_1 \geq cx$. Since $r_1(a,x,t)$ is piecewise linear in t, we see that we can assume without loss of generality that t_1 and t_2 are among $(a+1)x+a+1$, $(a+1)x+a$, $(a+1)x+1$, $ax+a+1$, $(a+1)x$, $x+a+1$, $cx+c$, cx. By Lemmas 3.13, 3.15, 3.14, 3.16, we see that t_1 can't be one of $(a+1)x$, $x+a+1$, $cx+c$, cx, and by Lemma 3.4 we see that t_2 can't be any of $(a+1)x+a+1$, $(a+1)x+a$, $(a+1)x+1$, $ax+a+1$. Since $t_1 < t_2$, we must have either $t_1 = ax+a+1$ or $t_1 = (a+1)x+1$, and $t_2 \neq x+a+1$.

If $t_1 = (a+1)x+1$, then we must have $t_2 = cx+c$, and $x \le \frac{c-1}{a+1-c}$. By Lemma 3.5 it's enough to check that $\frac{c-1}{a+1-c} \le \frac{2c-a-1}{2a+1-2c}$. This follows from

$$(2c-a-1)(a+1-c)-(2a+1-2c)(c-1) = (c-a)a \ge 0.$$

If $t_1 = ax+a+1$ and $t_2 = cx$, then $x \ge \frac{a+1}{c-a} = (c+a)(a+1)$. By Lemma 3.10, it's enough to check that $(c+a)(a+1)$ is bigger than both $c+a-1$ and $\frac{2a+2-c}{2c-a-1}$. The first of those is obvious, and the second follows from the fact that $(c+a)(a+1) \ge 2+2\sqrt{2} \ge \frac{2+3\sqrt{2}}{2}$, by the proof of Lemma 3.12.

If $t_1 = ax+a+1$ and $t_2 = cx+c$, then $x \ge \frac{a+1-c}{c-a} = c+a-1$, and we are done by Lemma 3.5. If $t_1 = ax+a+1$ and $t_2 = (a+1)x$, then we have $x \ge a+1$, so by Lemma 3.9 we are done.

Thus $t_1 < cx$, so we can assume without loss of generality that t_1 is one of $x-a, ax-1, x+a+1$. If $t_1 = x-a$, then by Lemma 3.12 we may take $t_1 = ax-1$ instead. Then by Lemma 3.16 and Lemma 3.17 we see that $t_2 \ge cx$ and $\frac{2a+2-c}{2c-a-1} \le x \le c+a-1$. □

We are finally ready for the main event.

Theorem 3.19. $r_3(a,x,t) \ge 0$.

Proof. It is enough to show that there is no $0 \le t_1 \le t_2 \le t_3$ such that $r_2(a,x,t_1) > 0$, $r_2(a,x,t_2) < 0$, and $r_2(a,x,t_3) > 0$: we have $r_3(a,x,0) \ge 0$ by Lemma 3.1, and we know $r_3(a,x,+\infty) = 0$, so thinking of $r_3(a,x,t)$ as a function of t, if it were ever negative for some t_* there would have to be a $0 \le t_1 \le t_*$ at which r_3 was decreasing, and a $t_2 \ge t_*$ at which r_3 was increasing. Furthermore, we have $r_2(a,x,(a+1)x+a+1-\epsilon) > 0$ for $0 < \epsilon < \frac{1}{2}$, so if we found t_1, t_2 we could find t_3 as well. So suppose, for the sake of contradiction, that such t_1, t_2, t_3 existed.

Using a similar argument to the above along with the fact that $r_2(a,x,0) = 0$, we can show that there must be s_0, s_1, s_2, s_3 such that $0 < s_0 < t_1 < s_1 < t_2 < s_2 < t_3 < s_3$, $r_1(a,x,s_0) < 0$, $r_1(a,x,s_1) > 0$, $r_1(a,x,s_2) < 0$, $r_1(a,x,s_3) > 0$.

By Lemma 3.11, we have $s_1 > a+1$. Thus by Lemma 3.18, we have $s_1 < cx$, $s_2 > \max(x+a+1, ax-1)$, and

$$\frac{2a+2-c}{2c-a-1} \le x \le c+a-1.$$

Note that the upper bound on s_1 implies that we may assume $t_2 \le cx$. Now we can apply Lemma 3.3 to see that we have $r_2(a,x,cx) \ge x$. Let s_2^* be the infimum of all possible choices of s_2. If $s_2^* > cx$ we have a contradiction. Otherwise, we have

$$1 \le x \le r_2(a,x,cx) < \int_{s_2^*}^{cx} -2r_1(a,x,t)dt,$$

and $r_1(a,x,t)$ is linear and decreasing on the interval $s_2^* \le t \le cx$. We have $r_1(a,x,s_2^*) = 0$, and by Lemma 3.10 we have $r_1(a,x,cx) > -1$, and we see that the average value of $-2r_1(a,x,t)$ along the interval is less than 1. Finally, we know that $\frac{\partial}{\partial t} r_1(a,x,t)$ is a negative integer along this interval, so the length of the interval is at most 1, and we see $\int_{s_2^*}^{cx} -2r_1(a,x,t)dt < 1$, a contradiction. □